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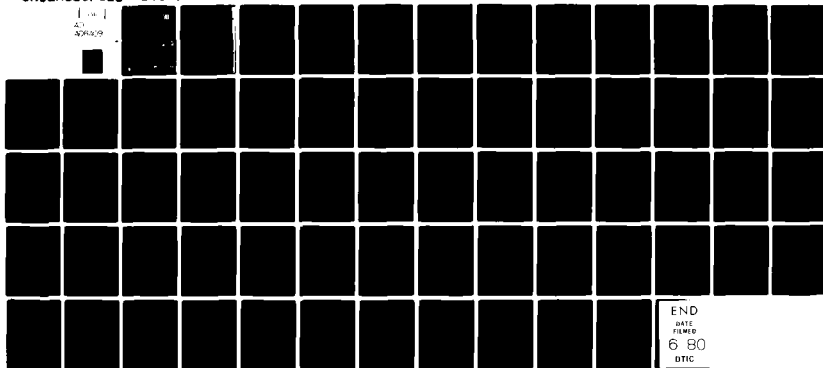   OCT 79 H MORITZ   F19628-79-C-0075
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CONCEPTS IN GEODETIC
REFERENCE FRAMES

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The report reviews conceptual aspects and problems arising in a precise definition of reference frames for the purpose of geodesy and geodynamics. The concepts: physical, conventional, and average system are introduced. Inertial systems are discussed from the point of view of classical mechanics and of the theory of relativity. The concept of celestial pole, which plays a fundamental role in a precise definition of precession, nutation and polar motion, is reviewed in →next page

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some detail. A section is devoted to tidal effects, which should be removed from the coordinates. Finally, various definitions of terrestrial reference systems are compared.

FOREWORD

CONTENTS

1. Introduction

The present report deals with a review of conceptual problems which are of importance for a definition of precise reference systems to be used for very accurate geodetic purposes, down to the centimeter level (1 part in 10^8 or better).

This subject has recently found great interest. The IAU Colloquium No. 26 on Reference Coordinate Systems for Earth Dynamics in Torun, Poland, 26-31 August, 1974, was the first international meeting devoted entirely to this topic. The papers collected in the Proceedings of this meeting (Kolaczek and Weiffenbach, 1975) form a basic source of reference for the present report. A particularly difficult and controversial subject, important in view of the recent discussions and resolutions of the IAU at its General Assemblies in Grenoble (1976) and Montreal (1979) on the topic of astronomical nutation, is the proper definition of the celestial pole. Basic recent references for this problem are (Leick, 1978), (Leick and Mueller, 1979), (Kinoshita et al., 1979), and (Fedorov, 1979).

The review article (Kovalesky, 1979) has been found helpful in preparing this report.

Particular thanks are due to Ivan I. Mueller for clarifying discussions and for providing relevant literature.

2. Physical, Conventional, and Average Systems

Let us start with a simple example familiar from plane surveying (Fig. 2.1). Let a number of points P_1, P_2, ... P_n be given in the plane. Various choices of a rectangular xy coordinate system are possible.

i) _Physical Systems._ -- Origin and direction of the coordinate axes are realized physically. For instance, the origin O may coincide with P_1 and the x-axis be defined so as to pass through point P_2 . (Fig. 2.2).
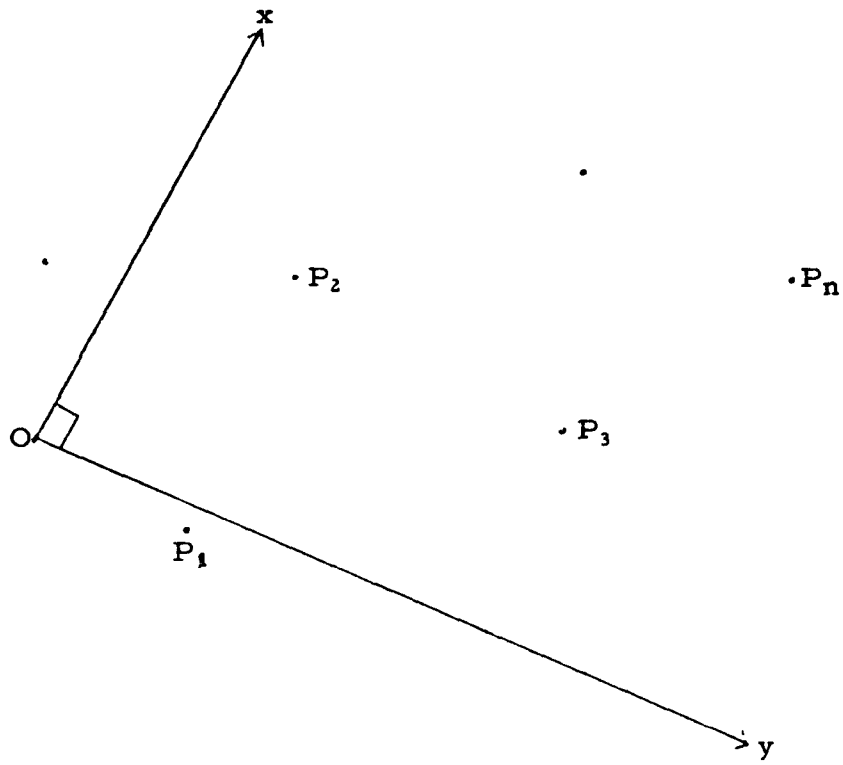
Figure 2.1 :   A plane coordinate system
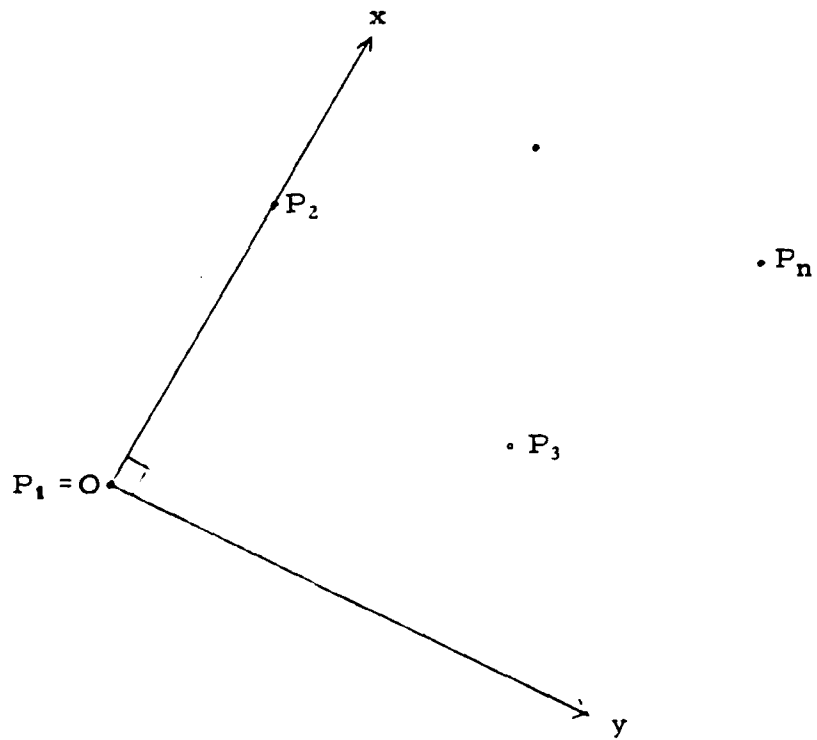
Figure 2.2:   A physically defined system

Another choice of a physically defined coordinate system would be the following: the origin O coincides again with a given point P_1, but the x-direction is realized by magnetic north as given by a magnetic compass (Fig. 2.3). If the direction of magnetic north differs from point to point, then we may select as x-axis the direction of magnetic north at P_1.
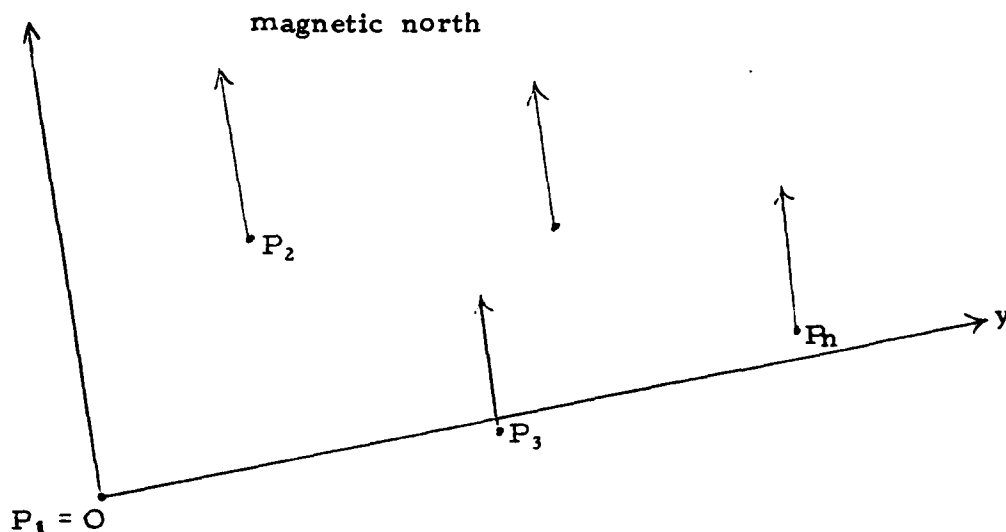
Figure 2.3 : Another physical definition

ii) <u>Conventional Systems</u>. -- In plane surveying, one frequently works with a coordinate system which is arbitrarily situated with respect to the points under consideration and whose axes has an arbitrary direction, such as shown in Fig. 2.1. Such a system is, for instance, defined by assigning to P_1 arbitrary coordinates (x_1, y_1) and to the line P_1P_2 an arbitrary direction angle.

iii) <u>Average Systems</u>. -- In choosing P_1 as the origin, we may seem to show undue preference to one of the n given points. A procedure for treating all points equitably consists in placing the coordinate origin at the center of mass S of the n points:

$$S: \quad \bar{x} = \sum_{1}^{n} x_i \; , \qquad \bar{y} = \sum_{1}^{n} y_i \; , \tag{2-1}$$

so that $\bar{x} = O$, $\bar{y} = O$ by definition. This determines only the origin; the x-axis may still be chosen conventionally or physically. As an example, assume that the direction of magnetic north D differs at the various points P_i ; then the x-axis could be chosen as the average magnetic north:

$$\bar{D} = \frac{1}{n} \sum_{1}^{n} D_i \; . \tag{2-2}$$

In the simple case in which the position of the points P_1, ..., P_n does not change with time, everything is obvious and the distinction between physical, conventional, and average systems appears trivial and pedantic, the more so as there is no clear border line between these concepts and they frequently overlap.

Time variations. The situation becomes much more complicated if we consider a variation of the position of points P_i with time. Assume, for the moment, that in the course of time, each point P_i describes a certain curve (Fig. 2.4). After a certain time, the points P_1 and P_2 will have moved to P_1' and P_2', and a system xy, physically defined according to Fig. 2.2, will have moved to the position x'y'. Even if the remaining points P_3, ..., P_n did not change their position with time, their coordinates would still change because the coordinate system changes. This shows that such a purely physical definition of the coordinate system would be inappropriate in this situation.

Modeling of time variations. It may, however, happen that the paths of movement of the points P_i are known in some coordinate system xy; more precisely we know the relative position

$$s(t) - x^{\circ} = \Delta x(t), \qquad y(t) - y^{\circ} = \Delta y(P) \tag{2-3}$$
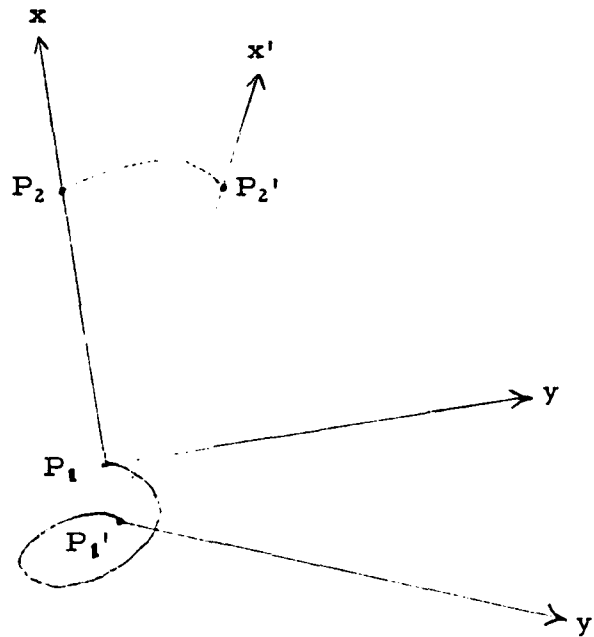
Figure 2.4: Time variations

of P_i with respect to some fixed "initial position" P_i^o of this point. Then the observed instantaneous position of any point can be reduced to this fixed position by subtracting $\Delta \underline{x}$:

$$\underline{x}^o = \underline{x}(t) - \Delta \underline{x}(t) \tag{2-4}$$

$\underline{x} = (x, y)$ representing the position vector.

These coordinates x, y refer to a system in which $\Delta \underline{x}(t)$ is given; it is thus a system which a priori given rather than operationally defined. Using our present terminology, it will be a conventional system.

Averaging of the residuals. It is realistic to assume that the point paths cannot be completely modeled; even after subtracting $\Delta \underline{x}(t)$ according to (2-4), the coordinates so obtained will not be completely time-independent, or in other terms, the residuals

$$\delta \underline{x}(t) = \underline{x}(t) - \Delta \underline{x}(t) - \underline{x}^o \tag{2-5}$$

will not be zero. We thus have

$$\underline{x}(t) = \underline{x}^{o} + \Delta\underline{x}(t) + \delta\underline{x}(t) , \qquad (2\text{-}6)$$

which is the basic decomposition of the real $\underline{x}(t)$ into:

1) a constant part $\underline{x}^{o}$ independent of time,

2) a functional part $\Delta\underline{x}(t)$ which can be modeled

    by a known functional expression, and

3) unknown residuals $\delta\underline{x}(t)$.

It can be hoped that $\Delta\underline{x}(t)$ models the time-dependent effect so well that the residuals will be small and irregular.

    Still, even after subtracting the functional effect $\Delta x(t)$, it may not be appropriate to select one of the points, say P_1, as the coordinate origin O, because residual motions of P_1 will then distort the coordinates of all the other points. If the residuals $\delta\underline{x}$ are really irregular without showing a trend, then it may be a good solution to place the origin O at the center of mass of the points P_i , using the definition (2-1) with $\underline{x}_i$ replaced by $\underline{x}_i - \Delta\underline{x}_i$. Then the system will be defined in such a way that it is little affected by random residual motions.

    <u>Physical versus conventional definitions again.</u>   To get a good intuitive understanding of the problem, let us consider another example. A geodetic reference ellipsoid should approximate the geoid reasonably well but it need not be the <u>best</u> approximation of the geoid by an ellipsoid. The latter ellipsoid would be the mean earth ellipsoid which is uniquely defined physically[1] (Heiskanen and Moritz, 1967, sec. 2-21). However, such an empirically determined mean earth ellipsoid would not be suitable as a reference ellipsoid for practical geodetic purposes, because its parameters change with every improvement of relevant measurements;

---

[1] Such a physical definition involves certain fundamental constants; it satisfies, however, also the condition that the average value of the geoidal height over the whole earth is zero; a curious interplay of "physical" and "average" aspects!

on the other hand, a practical reference ellipsoid should not be changed if possible because an enormous amount of data is based on it.

Therefore, a proper reference ellipsoid will be defined conventionally but in such a way that it is close to a physically defined mean earth ellipsoid.

Astronomical observations. As a final example, take astronomical observations, say time observations by means of a transit instrument (Mueller, 1969, p. 250). In the "ideal" case, the axis of the instrument should be horizontal, and the plane formed by successive positions of the line of sight should coincide with the meridian plane. Both "physically" defined conditions cannot be exactly met; rather than trying indefinitely to adjust the instrument to this "physical" position, the instrument is set up in an arbitrary, "conventional", position which is close to the physical one; the deviations from the latter are then taken into account by appropriate corrections. Again we may say that the reference axes for such observations are "conventional" but close to the "physical" position.

Preliminary remarks on a terrestrial reference system. For low accuracy requirements, we may consider the earth a rigid body which turns with constant angular velocity around the axis of maximum inertia. Then this axis of rotation would be a natural choice for the z-axis of a three-dimensional rectangular coordinate system xyz; the origin could be placed at the earth's center of mass; and a point rigidly connected to the earth's surface would define the xz-plane by requiring that this plane passes through the given point. This would be a proper physical definition.

The actual situation is enormously complicated by the fact that the earth is not rigid but has a shape that varies with time, and that the rotation axis differs from the axis of maximum inertia; neither the direction of the rotation axis nor the speed of rotation are constant.

In the sequel we shall discuss this problem at length. Here we remark only that a physical definition of the z-axis, either as the instantaneous rotation axis, as the axis of maximum inertia, or in some other way, is still theoretically possible, but such a choice would not be practically feasible. Therefore, one uses a <u>conventional</u> definition of the z-axis, presently the so-called Conventional International Origin (CIO). Similarly one does not use one observatory to define the zero meridian, but an <u>average</u> over a number of stations. The origin is at the <u>physically</u> defined center of mass of the earth.

Thus, practical terrestrial reference systems will again combine conventional, physical, and average aspects .

It may be asked (and has been asked) whether such a mixed definition is theoretically valid and does not possess internal inconsistencies. The answer is that, if everything is done properly, the system will be theoretically as "clean" as any other system. The deviations of the physical axes from their conventional counterparts can be taken into account by small corrections (such as also seen in the examples of a geodetic reference ellipsoid and astronomical instruments). Averaging is, of course, only to be introduced when the geometry is not violated, for instance for determining the zero meridian where there is only the question of fixing a constant rotation around the z-axis.

3. Inertial Systems -- Classical Theory

Assume that in a certain cartesian spatial coordinate system xyz, the equations of motion for a mass point have the Newtonian form:

$$\ddot{x} = X, \qquad \ddot{y} = Y, \qquad \ddot{z} = Z \qquad\qquad (3\text{-}1)$$

where dots indicate differentiation with respect to time:

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, \qquad \text{etc.} \qquad\qquad (3\text{-}2)$$

and X, Y, Z are the components of the external force in the xyz system.

Frequently it is more convenient to introduce <u>index notation</u> and write

$$x = x_1, \qquad y = x_2, \qquad z = x_3 \tag{3-3}$$

and similarly $X = X_1$, etc., so that (3-1) becomes simply

$$\ddot{x}_i = X_i \qquad (i = 1, 2, 3). \tag{3-4}$$

<u>A coordinate system in which the equations of motion have the</u> <u>Newtonian form (3-1) or (3-4) , is called an inertial system.</u>

If xyz is an inertial system, and if x'y'z' is another cartesian coordinate system which has a uniform translational motion (that is, without acceleration or rotation) with respect to the system xyz, then x'y'z' is likewise an inertial system.

Matters will be different as soon as the system x'y'z' accelerates or rotates with respect to the inertial system xyz: then the equations of motion will no longer have the simple form (3-1), and x'y'z' will not be an inertial system.

A spatial system xyz "fixed with respect to the stars" (this loose expression will be refined later) can be considered an inertial system; a system x'y'z' connected to the earth rotates relative to xyz and is thus not inertial.

The equations of motion for a rotating frame contain <u>rotational</u> <u>terms</u> such as the centrifugal "force" and the Coriolis "force" ; cf. (Synge and Griffith, 1942, p. 343). These terms are not real forces but only express the fact that the coordinate system is no longer inertial. Therefore, such fictitious "forces" are also called <u>inertial</u> <u>forces.</u> This terminology is well established although the expression "rotational terms" would be more appropriate.

The ephemerides of the planets and the moon are calculated by the integration of equations of motion of the form (3-1). Thus they refer to an inertial system. Inversely, observations of the planets or the moon in combination with calculated ephemerides can be used to establish an inertial system, as will be shown in Section 5.

Quasi-inertial systems.   Consider again a system x'y'z' moving
uniformly with respect to an inertial system xyz.   This means that
the displacement of x'y'z' with respect to xyz occurs with constant
speed and in such a way that the respective axes remain parallel to
each other; we may also speak of a uniform translation, which is a
parallel displacement along a straight line with constant velocity.   In
such a case, the system x'y'z' will again be inertial, as we have seen.

Consider now the case that the system x'y'z' undergoes a non-
uniform translation with respect to xyz.   This means that the movement
is such that the axes x'y'z' remain parallel to xyz but the origin O' of
x'y'z' may have a curvilinear accelerated motion with respect to the
system xyz (Fig. 3.1).   In this case, x'y'z' will no longer be an
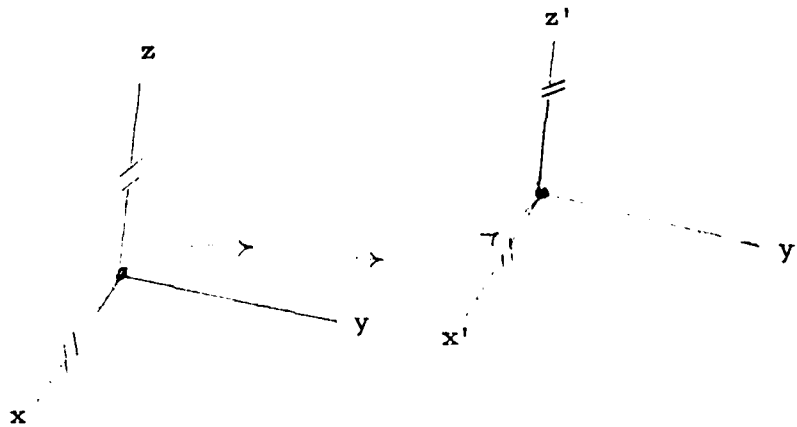inertial system.

Figure 3.1 :   A quasi-inertial system

However, it shares with an inertial system the property that
the direction of the coordinate axes does not change with time: the
system x'y'z' has no rotation with respect to the xyz frame.   Now in
astronomy generally it is the direction which really matters: there-
fore, for astronomical purposes the system x'y'z' behaves very much

like an inertial system; we may call it a quasi-inertial system.

An important example of such a quasi-inertial system is a system whose direction remains unchanged with respect to the stars and whose origin is at the earth's center of mass. Since the earth describes a nearly elliptic path around the sun, which is a curvilinear nonuniform motion, such a system is quasi-inertial rather than strictly inertial.

Axes which always remain parallel to a fixed direction can be realized by means of a gyroscope; the axis of an ideal drift-free gyroscope keeps its direction regardless of the motion of the gyroscope in space. Therefore, a coordinate system defined by gyroscopes is another physical realization of a quasi-inertial system. This is important in inertial positioning systems (Section 5).

Quasi-inertial systems are characterized by the fact that for them the equations of motion do not contain any rotational term. This does not yet mean, however, that they have the simple form (3-1): there will be additional terms which represent the effect of an acceleration of the coordinate system; cf. (Synge and Griffith, 1942, p. 344).

4. Inertial Systems -- Relativistic Aspects

A refinement of classical mechanics is provided by the special and the general theory of relativity. It turns out that for most practical purposes of geodesy, geodynamics, and space dynamics, classical mechanics is sufficient; if relativistic effects are relevant at all, then they can be taken into account by very small corrections.

It is, however, of great conceptual significance to understand the situation well also from the point of view of the theory of relativity. Since most textbooks on general relativity put more emphasis on other topics, it may be appropriate to review the impact of relativity on the problem of reference systems in some detail. The reader who is not interested in relativistic aspects may omit this section.

We shall mainly use the books (Misner et al., 1973) and (Ohanian, 1976), as well as the report (Moritz, 1967). The paper (Blais, 1978) also contains a brief treatment of relativistic aspects in reference frames.

Inertial systems in special relativity. In the special theory of relativity, inertial systems play a basic role as privileged coordinate system in space-time: in such a system, the four-dimensional line element has the simple form

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \qquad (4\text{-}1)$$

Here $x = x_1$, $y = x_2$, and $z = x_3$ denote rectangular coordinates in space, t designates the time, and c denotes the constant velocity of light in a vacuum; we have put $x_4 = ict$, where $i^2 = -1$. As in classical mechanics, a reference system moving with constant velocity with respect to an inertial system, is again an inertial system.

Transformations between inertial systems are such as to leave the line element (4-1) invariant (unchanged); the set of such "Lorentz transformations" form a group, the Lorentz group, which describes the symmetry of the space-time of special relativity.

No inertial systems in general relativity. The special theory of relativity holds only in the absence of a gravitational field. Gravitational fields are treated by the general theory of relativity. Here the line element has the form

$$ds^2 = \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}\, dx^\alpha dx^\beta , \qquad (4\text{-}2)$$

where x^α denotes coordinates x^1, x^2, x^3, x^4 in space-time, which will in general be curvilinear rather than rectangular. The $g_{\alpha\beta}$ are functions of these coordinates. Indices such as α and β run from 1 to 4; lower indices are called covariant, and upper indices, contravariant. The Einstein summation convention, which will be used in this section, prescribes summation with respect to any index that occurs in both an upper and a lower position, as shown in Eq. (4-2).

The coordinates x^α now have upper indices because the differentials dx^α form a "contravariant vector".

The line element (4-2) relates to (4-1) in much the same way as a line element on a curved surface,

$$ds^2 = Edu^2 + 2F\,dudv + Gdv^2, \tag{4-3}$$

relates to a line element in the plane,

$$ds^2 = dx^2 + dy^2 \tag{4-4}$$

Here u, v are curvilinear coordinates and E, F, G form the "metric tensor"

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}. \tag{4-5}$$

In space-time, the metric tensor $[g_{\alpha\beta}]$ is a 4x4 matrix, so that there is full analogy between the general forms (4-2) and (4-3) on the one hand, and between the "inertial forms" (4-1) and (4-4), on the other hand. In a way, the general theory of relativity is nothing else but an extension of the theory of two-dimensional surfaces to four-dimensional space-time.

This analogy will help understand an important point. On a curved surface one can introduce coordinate which, in an infinitesimal neighborhood of a point, give a line element

$$ds^2 = du^2 + dv^2 \tag{4-6}$$

which has the same form as the plane element (4-4). Geometrically, this means that, in a small neighborhood of this point, the surface is approximated by its tangent plane.

However, it will not be possible, in general, to introduce coordinates in such a way that the "inertial form" (4-6) holds on the whole surface (or even in a finite part of it).

Transferred to four dimensions, this reasoning shows that, in a curved space-time, it will be possible to introduce coordinates which

correspond to an inertial system in an infinitesimal neighborhood of a point; but it will be impossible to introduce an inertial system that is valid for the whole space-time.

In this sense, there are no inertial systems in general relativity. All possible coordinate systems are, in principle, equivalent; there are no privileged systems. This is Einstein's Principle of General Covariance, or General Relativity.

Another important principle in this theory is the Principle of Equivalence, according to which gravitational and inertial forces (such as the centrifugal or the Coriolis force) are basically identical: both are effects of a deviation of the coordinate system of line element (4-2) from an inertial system of line element (4-1). Thus gravitation is interpreted geometrically as an effect of the curvature of space-time.

Both the Principle of Equivalence and the Principle of General Covariance have played a fundamental heuristic role in Einstein's considerations leading to his theory of gravitation around 1915 bec ause these principles provide a natural transition from the flat space-time of special relativity to the curved space-time of general relativity. Einstein's heuristic procedure is still the best way for understanding this theory; hence it is strongly emphasized in almost every textbook on general relativity.

However, the relativistic treatment of reference systems requires some subtler distinctions which show that, after all, privileged systems can be introduced which serve as practically satisfactory approximations to inertial systems, both on a local and on a global level. There is also an analogue of the "quasi-inertial systems" introduced in the preceding section.

Local inertial systems. Just as a curved surface can be approximated locally by a tangent plane, so curved space-time can be approximated, in the neighborhood of a certain point, by a tangent "plane" space-time in which an inertial system can be introduced.

Thus, in a certain "small" region, inertial systems are possible even in general relativity. Since our space-time is only very slightly curved, the gravitational field in the solar system being very weak, the "small" region just mentioned certainly covers the solar system and even extends well beyond. According to Eddington (1924, pp. 99) a local inertial system will deviate from a global system by about 2 seconds of arc in a century.

Global nearly-inertial systems. The application of the relativistic theory of gravitation to the region of our solar system requires boundary conditions at infinity: with increasing distance from the attracting masses the effect of gravitation vanishes, and the curved space-time becomes flat at infinity. This fact permits the introduction of uniquely defined privileged systems, the harmonic coordinate systems. These systems rigorously refer to curved space time. At infinity they reduce to inertial systems of form (4-1), and within the solar system they approximate inertial systems practically very well.

In this sense, the harmonic coordinates form a privileged coordinate system, which is a natural generalization of an inertial system to curved space-time. This has been particularly emphasized by Fock (1959).

Quasi-inertial systems and Fermi propagation. In the last section we have introduced "quasi-inertial systems". They are three-dimensional cartesian systems whose origin is moving arbitrarily but whose axes remain always parallel; a physical realization is by means of axes whose direction is stabilized by means of gyroscopes. The underlying principle is that the axis of a freely spinning gyroscope maintains its direction even if its frame is accelerated or rotated; furthermore, the axis is unaffected by gravity.

This concept of a quasi-inertial frame can be defined also in general relativity. The relevant concept is Fermi propagation, or Fermi-Walker transport, which is considered in detail and used extensively

in (Synge, 1960). It is also treated in (Misner et al., 1973, p. 170), but hardly elsewhere in standard textbooks. Therefore we shall briefly consider it here, following (Moritz, 1967).

The equation of Fermi-Walker transport may be written

$$\frac{\delta \lambda^\alpha}{\delta s} = \lambda_\beta \left(\frac{\delta u^\beta}{\delta s} u^\alpha - \frac{\delta u^\alpha}{\delta s} u^\beta \right) \tag{4-7}$$

(Synge, 1960, 1. 13). Here λ^α (or λ_β) are the contravariant (or covariant) components of the vector undergoing Fermi propagation, related by

$$\lambda_\alpha = g_{\alpha\beta} \lambda^\beta. \tag{4-8}$$

The vector u^α is the four-velocity

$$u^\alpha = \frac{dx^\alpha}{ds} , \tag{4-9}$$

The unit vector of the tangent to the world line of the particle to which the vector λ^α is attached (Fig. 4.1). The symbol δ denotes covariant differentiation:

Figure 4.1 : Fermi-Walker transport

$$\frac{\delta\lambda^\alpha}{\delta s} = \frac{d\lambda^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\lambda^\beta u^\gamma \quad , \tag{4-10}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, and analogously for $\delta u^\alpha/\delta s$.

In our case, the vector λ^α represents the spin axis of the gyroscope. It lies in the instantaneous three-dimensional space of the spinning particle and is therefore orthogonal to u^α:

$$u^\alpha\lambda_\alpha = 0 \tag{4-11}$$

Hence (4-7) reduces to

$$\frac{\delta\lambda^\alpha}{\delta s} = \lambda_\beta \frac{\delta u^\beta}{\delta s} u^\alpha \quad . \tag{4-12}$$

This equation holds for Fermi-Walker transport of a space-like vector satisfying (4-11). It expresses the fact that the change $\delta\lambda^\alpha/\delta s$ has the direction of u^α and consequently has no component in the instantaneous three-space of the observer. Thus the change of λ^α is purely in time: the vector λ^α remains unchanged in space, it is transported parallelly. This shows that Fermi propagation is related to spatial parallelism.

Consider now a system of three mutually orthogonal space-like vectors λ^α, each of which is represented by the axis of a freely spinning gyroscope. In this way the axes of a rectangular xyz system which is transported parallelly in space, may be realized physically.

It can be shown (Moritz, 1967, p. 47) that the change $\delta\lambda^\alpha/\delta s$ is small of order c^{-2}, c being the velocity of light. To this accuracy, the direction of Fermi-propagated axes remains constant in space; it furthermore is practically unaffected by the gravitational field.

This shows that gyroscopically stabilized "quasi-inertial systems" are possible even in the context of general relativity.

Separation of gravitation and inertia. After this discussion of "privileged" coordinate systems which seem to contradict the Principle of General Covariance, let us now briefly remark on the separation of gravitational and inertial forces, which seems to violate the Principle of Equivalence.

This question is related to the problem of reference systems only indirectly; furthermore it has been dealt with rather fully in two reports (Moritz, 1967, 1971); therefore we shall be very brief.

The Principle of Equivalence states that, because of the identity of gravitational and inertial mass (shown experimentally by R. Eötvös around 1900 for an accuracy of 5×10^{-9} !) the resultant of gravitational and inertial forces acting at one point cannot be separated into a gravitational and an inertial part; both are equivalent and cannot be distinguished.

Matters are different if we consider, not only one point, but a region in space, which may be arbitrarily small. In the theory of surfaces, the Gaussian curvature K provides a criterion for distinguishing a curved surface from a plane, depending on whether K is nonzero or zero. The generalization of the Gaussian curvature to four dimensions is the Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$; again, space-time is flat if $R_{\alpha\beta\gamma\delta} = O$ and curved otherwise. Now, curvature of space-time is an objective criterion for the presence of a genuine gravitational field, so that, according to (Synge, 1960, p. 109), we may write

$$R_{\alpha\beta\gamma\delta} = \text{gravitational field} \tag{4-13}$$

The Riemann curvature thus provides a criterion for the presence of a gravitational field, but not yet a means for the separation of gravitational and inertial effects. In flat space-time, inertial forces have an objective significance since they are due to the deviation of the observer's coordinate system from an inertial system. Similarly in a

weak gravitational field, a separation of gravitation and inertia is feasible if we succeed in introducing a privileged coordinate system similar to an inertial system. In this way, the separation of gravitation and inertia is intimately connected with the question of an "almost" inertial reference system, such as the harmonic system mentioned above.

We finally point out that in such a system there is approximately (Moritz, 1967, p. 43)

$$c^2 R_{i4j4} = \frac{\partial^2 V}{\partial x_i \, \partial x_j} \qquad (4\text{-}14)$$

where i and j are spatial indices running from 1 to 3. Thus, second-order gradients of the potential V are purely gravitational. In (Moritz, 1971) we have shown that using a combination of accelerometers, measuring first-order gradients, and gradiometers, measuring second-order gradients, a separation of the gravitational signal from inertial disturbances can be effected even with first-order gradients, that is, in the gravitational force.

Cosmological questions. For a homogeneous and isotropic universe, the line element (4-2) has the form (Bondi, 1960, p. 102):

$$ds^2 = dt^2 - [R(t)]^2 \; \frac{dx^2 + dy^2 + dz^2}{[1 + (k/4)(x^2 + y^2 + z^2)]^2} \qquad (4\text{-}15)$$

Here R(t) is a time-dependent scale factor by means of which the expansion of the universe can be described. The constant k may have the values +1, 0, or -1. For k = O, space is Euclidean; for k =1, space has constant positive curvature, and for k = -1, constant negative curvature. The space-time described by (4-15) is called the Robertson-Walker model.[1]

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[1] For k = O and R = c^{-1} , (4-15) reduces for (4-1), apart from the irrelevant factor $(-c^2)$.

This model appears well suited to describe mathematically the large-scale space-time structure of the universe, apart, of course, from "local" gravitational irregularities such as caused by our solar system.

On the basis of present observational data it is not possible to decide clearly whether k is positive, negative or zero, although there is some indication that space may have negative curvature, cf. (Ohanian, 1976, p. 416).

At any rate, Robertson-Walker space-time will not in general be the flat space-time of special relativity (4-1). Thus, strictly speaking, inertial systems in the usual sense will not exist. This leads us to the following paradox: The most accurate means of practically establishing an inertial system is VLBI using quasars; however, quasars are a typical phenomenon of an expanding universe which is described by the curved space-time (4-15) <u>for which no inertial systems exist!</u>

However, this paradox is a theoretical curiosity rather than a fact of practical significance. Indeed, as we have seen above, all our practical inertial systems are nonrigorous in the sense of general relativity but still perfectly useful. For the region of our galaxy, we may easily consider space-time to be essentially flat, apart from local gravitational irregularities. The same holds <u>a forteriori</u> for our solar system. Furthermore it is possible to study cosmology within the frame of special relativity and even of classical mechanics (Bondi, 1960, chapters XI and IX).

<u>Relativistic effects.</u> (Above we have seen that rotation is an "absolute" phenomenon in "general relativity"[1], being almost the same as in classical mechanics. Geodesics are not absolute in this sense

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[1] In fact, Fock (1959) and Synge (1960) have pointed out that this name is actually a misnomer; the name "geometrodynamics" (J. A. Wheeler) appears to be more appropriate.

since planetary orbits are geodesics in general relativity but ellipses (and not straight lines which would be geodesics) in classical mechanics. However, null geodesics representing light paths do appear as three-dimensional straight lines (to order V/c^2); and also time, to the same order, is identical to the time in classical mechanics. The same holds for light velocity in a vacuum.

Thus, the concepts: straight line (as represented by a light ray), time, light velocity, and rotation are practically the same in general relativity as in classical geometry and mechanics. Hence, to an accuracy of V/c^2 , the definition of reference systems (which is done in terms of these concepts) is practically the same as in classical mechanics.

What is the order of magnitude of V/c^2? The gravity potential W at the surface of the earth is approximately

$$W = 6.3 \times 10^7 \ m^2 s^{-2} \quad ;$$

cf. the value U_0 given in (Heiskanen and Moritz, 1967, p. 80); for the present purpose, the gravitational potential V and the gravity potential W (including the centrifugal force) are nearly equal. Then

$$\frac{V}{c^2} \doteq \frac{W}{c^2} \doteq \frac{6.3 \times 10^7 \ m^2 s^{-2}}{(3 \times 10^8 \ ms^{-1})^2} = 0.7 \times 10^{-9} \qquad (4\text{-}16)$$

Thus we may say that, in the neighborhood of the earth,

$$\text{general-relativistic effects} \doteq 10^{-9} \qquad (4\text{-}17)$$

In general, this will be below the level of 10^{-8} which can reasonably be expected for high-precision geodetic and geodynamic work in the near future.

So much for general-relativistic effects related to the presence of a gravitational field. There are also effects of special relativity which become significant for high velocities v. They have the order $(v/c)^2$. Assume

$$v = 29.8 \text{ km s}^{-1} = 10^{-4} c \qquad (4-18)$$

which is the velocity of the earth in its orbit around the sun. Then

$$\left(\frac{v}{c}\right)^2 = 10^{-8} \qquad (4-19)$$

which is of a similar order of smallness as (4-17).

Thus, relativistic effects are so small that it is difficult to measure them. Some of them furnish experimental tests of the general theory of relativity and are then well covered in the textbook literature. We shall, therefore, limit ourselves to a few brief remarks.

Time. Since time can be measured by means of atomic clocks far more accurately (of order 10^{-13} or better) than any other relevant quantity, relativistic effects can be shown here quite well. Since the proper time τ of the clock is related to coordinate time t by

$$d\tau = \left(1 - \frac{W}{c^2}\right) dt \qquad (4-20)$$

we have

$$\frac{dt - d\tau}{dt} = \frac{W}{c^2} \doteq 0.7 \times 10^{-9} \quad . \qquad (4-21)$$

Thus atomic clocks depend on the potential in a similar way as the old pendulum clocks depended on gravity, through incomparably less. Just as gravity, or better gravity differences, can be measured by means of pendulums, so the potential, or better potential differences, can, in principle, be measured by atomic clocks.[1]

Of such nature is the experimental by Pound and Rebka described in (Misner et al., 1973, p. 1057) and (Ohanian, 1976, p. 212), which

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[1] It would, however, be premature to hope for a new geodetic instrument measuring potential differences in this way: 1 cm in elevation would correspond to 10^{-18} in time!

measures the gravitational redshift of γ-rays using the Mössbauer effect and hence the potential difference. (Redshift occurs if the "clock" represented by the emitting source is slower.)

Thus, highly precise atomic clock readings must be reduced by the factor W/c^2 to refer them to a common standard.

Related phenomena are the time delay of radar echos from Mercury, Venus, and Mars due to their gravitational fields as measured by Shapiro and others (Ohanian, 1976, p. 128), and time dilation experiments measuring the redshift of different spectral lines of the sun and other stars (ibid., p. 214).

Another question is the relation between Atomic Time (AT) and Ephemeris Time (ET). Conceptually, AT is the time of quantum theory, and ET is the time of mechanics (classical or relativistic). If General Relativity is correct, then AT = ET. On the other hand, (Duncombe et al., 1975, p. 232) state that empirical observations tend to indicate that these two time scales are not equivalent. As an explanation they suggest that the gravitational constant G decreases at a rate of about 10^{-10} per year. This would mean that even Einstein's theory of gravitation would have to be generalized (theories of Jordan and Brans-Dicke); cf. (Misner et al., 1973, p. 1070) and (Ohanian, 1976, pp. 187-188).

Length. The present definition of the meter in terms of a certain multiple of the orange line of krypton will probably be given up in the near future. It will be redefined in terms of the atomic second and the velocity of light, of which the presently accepted value is

$$c = (299\ 792\ 458 \pm 1.2)\ \text{ms}^{-1} \tag{4-22}$$

(Moritz, 1975); this indirect definition of length will be more accurate.

In fact, since c is accurate to about 4 parts in 10^9, the new definition of length will be as accurate (time being defined with superior precision). Relativistic effects are below this level, so that the

influence of these effects on length will be negligible still for some time.

Light rays. Light rays can be regarded as straight except under unusual circumstances. Classical is the deviation of a light ray grazing the sun during an occultation. Modern results concerning this phenomenon and concerning analogous deflections of radio waves are given in (Ohanian, 1976, pp. 124-125); the order of magnitude is 1-2 seconds of arc.

Gyroscopic effects. Above we have seen that gyroscopes undergoing Fermi-Walker transport behave very much as in classical mechanics. Small relativistic effects ("geodetic precession") are described in (Ohanian, 1976, pp. 292-298).

Influences on planetary motion. The classical example is the precession of the perihelion of the orbit of the planet Mercury (about 40" per century). There are also periodic relativistic effects in earth-moon separation on the order of 1 m, which can be measured by lunar lasar ranging (Misner et al., 1973, p. 1048).

5. Inertial Systems -- Practical Realization

Inertial systems can be realized either geometrically or dynamically. In the geometrical realization, one uses objects which are considered at rest in the inertial system under consideration. In the dynamical realization, one uses objects (satellites, the moon, or planets) which move in a way described by the equations of motion in the inertial system to be realized.

The first statement is oversimplified and must be made more precise. In astronomy one is mainly interested in directions, so celestial objects (stars or quasars) can be used to define directions even if they are not at rest but move only along the direction under consideration. In other words, their motion should have only a radial but no transversal component.

The following review is primarily based on the papers collected in (Kolaczek and Weiffenbach, 1975); the article (Kovalevsky, 1979) has been found useful.

Stellar systems. Such systems are the best known since the star coordinates (right ascension α and declination δ) tabulated in astronomical catalogues are based on some stellar systems. The most accurate stellar systems are FK4 and FK5, the latter being elaborated now (Fricke, 1975). A detailed review is found in Chapter 6 of (Mueller, 1969).

Right ascension α and declination δ refer to an equatorial coordinate system for which the z axis has the direction of the earth's axis of rotation (the celestial pole, a precise definition of which will be found in Section 7). Since the celestial pole changes due to the effect of precession and nutation, the tabulated coordinates refer to some epoch t_0.

This coordinate system is easily accessible through observations of stars and is, therefore, used in geodetic astronomy, photo-grammetric satellite triangulation, etc.

However, the use of stellar systems is limited for various reasons; imperfectly known proper motions of stars, errors in the formulas for precession and nutation, etc. According to (Mueller, 1969, p. 197) and (Fricke, 1975, p. 216) the standard errors in right ascension and declination of the FK4 system, referred to the initial epoch, are about $\pm 0.014"$ on the equator and increase to $\pm 0.017"$ in the northern and to $\pm 0.04"$ in the southern hemisphere. These errors represent only the deviation of the FK4 from an ideal system; they do not include proper motion errors etc.

An important phenomenon is the occurrence of residual rotations of the stellar system. Any system based on the stars of our galaxy will contain a rotation if the galaxy rotates as a whole with respect to distant galaxies. To determine such a residual rotation, one may

either incorporate observations to visible extragalactic sources (quasars) or observations of planets, thus combining the celestial system with an extragalactic radio system or a dynamically defined inertial system (see below). This leads to very difficult problems discussed in the papers (Fricke, 1975) and (Duncombe et al., 1975).

Thus the long-term precision of a stellar system is limited to about ± 0.1" per century.

In order to improve the accuracy of a stellar reference system, observations in an astrometric satellite have been proposed by Bacchus and Lacroute. It is hoped that these observations will determine star positions to ± 0.01" and eventually to ± 0.001".

Extragalactic radio system. Such a system is defined by positions of a number (perhaps 20 or more) of extragalactic radio sources (quasars) whose relative positions are measured by very-long-base-line interferometry (VLBI). These sources have large radial motions due to the expansion of the universe but no observable trans-versal motions (since quasars are so far away, transversal motions would have to be enormous in order to be measurable).

In this way one can obtain a self-contained highly precise refer-ence systems, which is independent of the classical stellar systems but can be related to them as mentioned above. The accuracy of such a VLBI-determined reference system is eventually expected to be on the order of ± 0.001" and thus better than the FK 4 by at least one order of magnitude. Furthermore, the system is independent of the position of the earth's axis and can thus be used to determine precession and nutation, as well as polar motion and earth rotation (UT1); cf. (Shapiro, 1978).

After this review of geometrically defined reference systems (stellar and extragalactic system) we come now to the discussion of

dynamically defined systems which essentially[1] use the equations of motion of planets, the moon or artificial satellites.

Classical observations to the planets and the moon are used to determine Ephemeris Time (Section 4) and the rotation of the celestial system (see above) rather than for defining an independent reference system. For the latter purposes one uses lunar laser ranging and precise satellite techniques.

Lunar laser ranging. The motion of the moon in its orbit, as given by lunar ephemerides, defines an inertial system. It is a dynamical system as the orbital ephemerides have been computed using celestial mechanics.

If only laser distances (without reference to the star background) are measured, then this system is "blind" or self-contained in the sense that it can be used only for lunar laser ranging. It can, however, be related to the FK 4 or a similar stellar system using meridian observations and occultations of the moon, but with a lesser precision (Mulholland, 1975).

For modelling the motion of the moon, the orbital motion of its center of mass and the rotation (libration) of the moon is required. The parameters of these movements are themselves gradually improved by the lunar laser ranging observations. Eventually, modelling should be possible to the centimeter level, but there may be a residual long-term drift due to the secular acceleration of the moon on the order of 5" per century squared.

This system is excellent for periods up to say 10 years. A comparison on combination with VLBI determined extragalactic systems appears particularly promising.

Satellite ranging. The same principle of a self-contained dynamical system may be applied to satellite tracking (laser and doppler).

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[1] It may be argued that celestial systems also contain a dynamical component since precession and nutation are based on the theory of motion of the earth.

Unfortunately, satellite orbits cannot be as accurately modeled as the lunar orbit due to imperfect knowledge of the gravitational field, radiation pressure, and other effects. Especially the long-term stability is far less than in the case of the moon. The advantage of doppler systems is the ease with which continuous observations can be performed.

Such systems have proven useful for the study of short-period phenomena, especially of polar motion and earth rotation.

Inertial positioning systems. Directions that are constant in an inertial system (or change in a prescribed way with respect to such a system) can be realized by means of gyroscopes. Position differences with respect to such a system are obtained by twice integrating accelerations measured by means of accelerometers. Gyroscopically defined directions are practically independent on the gravitational field, but accelerations depend strongly on gravity in view of the Principle of Equivalence which expresses the impossibility of separating gravitational and inertial accelerations (Section 4). On the other hand, such a separation can be performed in second-order gradients, which provides a means for improving the accuracy of inertial positioning systems through gradiometry.

Present accuracies (a few decimeters for distances up to 50 km) are of considerable interest in geodetic surveying work, as has been pointed out at the First International Symposium on Inertial Technology for Surveying and Geodesy in Ottawa, October 1977. They are not yet suitable, however, to define an inertial reference frame of an accuracy considered in this report. Inertial positioning systems possess, however, great potentialities of development and of increase in accuracy by several orders of magnitude (Draper, 1977).

6. Celestial and Terrestrial Systems

So far we have considered inertial, or stellar, or celestial systems which are, roughly speaking, at rest with respect to the stars.

Let us now introduce a _terrestrial system_ which, also roughly speaking, is at rest with respect to the earth. These two systems differ primarily by the earth's rotation, but there are also other small rotations due to precession, nutation, and polar motion.

We shall start with a rather crude definition of a terrestrial coordinate system xyz, which will later be made more precise. The origin is at the _geocenter,_ the earth's center of mass. The most natural choice for the z-axis would be the earth's axis of rotation but, because of _polar motion,_ the rotation axis slightly changes its position within the earth's body. Figure 6.1 shows polar motion as projected
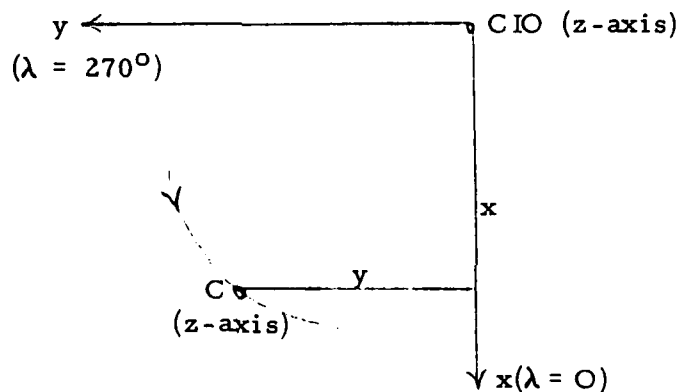
Figure 6.1 : Polar motion

onto a tangent plane at the North Pole of the earth. The celestial pole, which here represents the instantaneous rotation axis (in the next section, we shall make a distinction between these two concepts) describes a roughly circular path. The origin CIO (Conventional International Origin) corresponds to the average position of C in the years 1900-1905. The period of C around the origin is about 1.2 years, the Chandler period.

The z-axis of the terrestrial coordinate system is defined such as to pass through CIO; thus it corresponds to an average rather than instantaneous rotation axis. Not only the z-axis is conventional but

also the x-axis corresponding to the zero meridian ($\lambda = O$; φ and λ are geographical latitude and longitude, respectively). This zero meridian corresponds roughly to the Greenwich meridian; however, it is implicitly defined through a computational procedure adopted by the Bureau International de l'Heure (BIH). It is thus called the BIH zero meridian.

The y axis is directed for $\lambda = 90^{\circ}$ so that xyz forms a right-handed system. It should be noted that the conventional expression of polar motion in terms of the pole coordinates x, y (Fig. 6.1) is in disagreement with this global terrestrial system since there the y-axis is directed towards $\lambda = 270^{\circ}$. Thus the y-axis for polar motion has a direction opposite to the global y-axis; the x-axes are parallel. Furthermore, as we have said, x and y for polar motion are expressed in seconds of arc rather than in meters.

The stellar system XYZ is defined as follows. The origin may, for instance, also coincide with the geocenter (it is then quasi-inertial rather than inertial, cf. Section 3). The Z-axis is directed along the instantaneous rotation axis (more precisely, the celestial pole C, see next section); cf. Fig. 6.1. The X-axis has the direction of the vernal equinox, and Y is directed so as to form again a right-handed system.

Since the direction of the rotation axis in space changes because of precession and nutation (secular and periodic parts, respectively), the instantaneous celestial system so defined rotates very slowly but continuously with respect to inertial space and is thus not an inertial system. However, the system XY Z referred to a certain epoch to (say 1979, O) is inertial (apart from galactic rotation, see Section 5).

The transformation from the celestial system XYZ at an epoch to the terrestrial system xyz at epoch t is effected as follows. Since the common origin is the geocenter, both systems are transformed into each other by a rotation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad (6-1)$$

The rotation matrix A may be split up as the product of 4 rotations:

$$A = WRNP \qquad (6-2)$$

which have the following meaning.[1]

The matrices P and N express the effect of <u>precession and nutation</u>, respectively. The result of multiplying the vector (X, Y, Z) by the product NP is to transform the system from the initial epoch t_0 to the actual epoch t under consideration. Initially, the Z-axes coincided with the (mean) celestial pole at time t_0, after transformation by NP, it coincides with the (true) celestial pole at time t. Also the X-axis, passing through the vernal equinox, changes correspondingly.

Expressions for P and N can be found in (Mueller, 1969, pp. 62-79). The problem of the celestial pole and precession and nutation will be taken up again in a more precise way in the following section.

The matrix R expresses the main motion of the earth-fixed system xyz with respect to the space-fixed system XYZ: the <u>earth's rotation</u>. It has the form

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (6-3)$$

where θ is Greenwich Apparent Sidereal Time (GAST); cf. (Mueller, 1969, p. 139). It is related in a simple way to Universal Time, UT more precisely to UTO which is not yet corrected for polar motion (<u>ibid.</u>, p. 164) note that, before as well as after this transformation,

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[1] A general geometric treatment of transformations between reference frames is found in (Grafarend et al., 1978).

the z-axis is the instantaneous rotation axis.

A final transformation is effected through multiplication by the matrix W which expresses the effect of polar motion, also called polar wobble. It has the form

$$W = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & -y \\ -x & y & 1 \end{bmatrix} \qquad (6-4)$$

where x, y are the coordinates of the pole in the system of Fig. 6.1.

Precession and nutation are regular enough to be fairly accurately modelled by analytical expressions with coefficients that are partly empirical and partly come from theory; see Section 7. Polar motion is so irregular that it can only be determined from observations. In view of small irregularities of the earth's rotation, the same holds here, too. This will now be briefly reviewed.

Polar motion and earth rotation. Both phenomena are closely related. The earth's rotation can be expressed by means of the rotation vector $\underline{\omega}$ which has the direction of the rotation axis and the length ω which is the angular velocity of the earth's rotation. Then polar motion expresses the variation in the direction of $\underline{\omega}$ and irregularities in the earth's rotational velocity express the variation of the length of $\underline{\omega}$.

The classical book on this subject is (Munk and MacDonald, 1960). Later developments are reviewed in (Rochester, 1973) and (Lambeck, 1978). All these authors primarily discuss geophysical aspects.

Polar motion has been determined by the International Latitude Service (ILS) established in 1899 and expanded in 1962 as the International Polar Motion Service (IPMS). ILS comprises 5 stations distributed along the parallel $39°08'$, which have been continuously operating since 1899 until now since they also form part of IPMS (which comprises some 75 instruments).

Modern techniques for determining time and polar motion are doppler and laser ranging to artificial satellites and to the moon, and VLBI. Present accuracy in the pole position is on the order of 1m (0.03") (Capitaine and Feissel, 1975): modern techniques give 0.01" (Carter, 1978; Robertson et al., 1979) and are expected eventually to provide 0.002" in pole position and 0.1 msec in UT (Aardom, 1978).

The continuing importance of the ILS stations rests in the continuity of the observations of polar motions since 1899; the Conventional Internation Origin (CIO) for polar motion is based on these observations.

A review of polar motion services is (Guinot, 1978); many details can be found in papers collected in (Kolaczek and Weiffenbach, 1975). A recent reference on theoretical problems of the earth's rotation is (Guinot, 1979).

7. The Celestial Pole

In the preceding section we have only distinguished the instantaneous rotation axis and a mean rotation axis such as defined by CIO, and have identified the Celestial Pole, to which precession and nutation refer, with the instantaneous rotation axis. For higher accuracies this is no longer satisfactory and more precise distinctions must be made, according to developments made within the last two years and leading to a new system for precession and nutation adopted by the International Astronomical Union at its General Assembly in Montreal in August 1979.

A very clear presentation of the principles is found in (Leick, 1978) and (Leick and Mueller, 1979). We can here only describe the basic results and refer the reader for details to this work.

Formerly a rigid earth model formed the basis of a theory of precession and nutation (Woolard, 1952). The earth model on which the IAU resolution of 1979 is based is elastic with a liquid core. As an introduction we shall first discuss the case of a rigid earth and
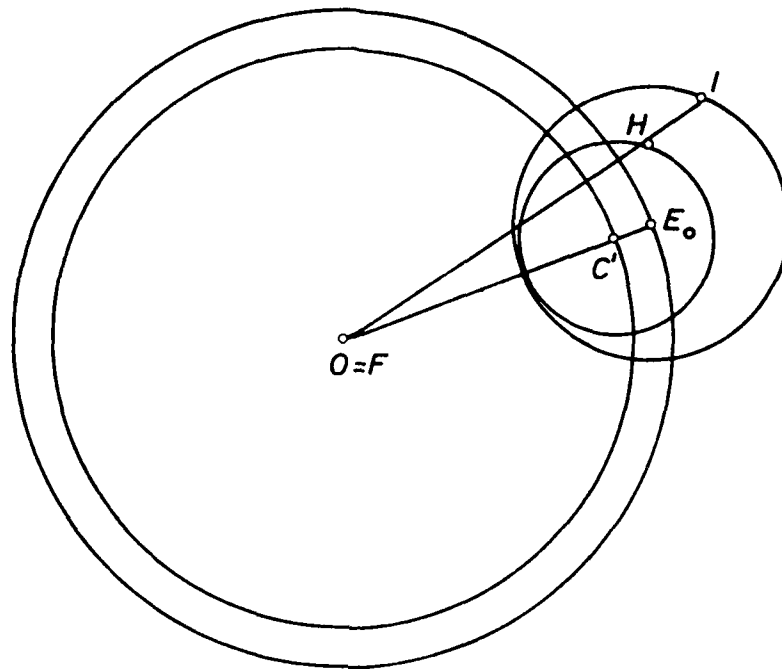
Figure 7.1 : Body-referred motions for a rigid earth

then the more complicated elastic earth model.

    Rigid earth.   Let us look at the earth in the vicinity of the
North Pole from above, similarly as in Fig. 6.1), see Fig. 7.1.
We shall use the following abbreviations:

F = figure axis,

I = instantaneous rotation axis,

H = angular momentum axis,

E_0 = Eulerian pole of rotation,

C' = celestial pole.

The figure axis F (we do not distinguish between an axis and its pole
which is its intersection with the celestial or terrestrial sphere) is
the axis of maximum inertia; if the earth were a precise ellipsoid of
revolution, then F would denote its axis of symmetry.

The axis H corresponds to the direction of the angular momentum which plays a basic role in the dynamics of a rigid body, as well known from mechanics (cf. Synge and Griffith, 1942, Chapter 14).

The poles E_0 and C' are explained as follows. If there were no external forces acting on the earth, then $C' = H$ and $E_0 = I$, and C' and E_0 would describe circles around F as shown in Fig. 7.1. The outer circle represents the motion of the instantaneous pole (in the force-free case coinciding with E_0) around the figure axis F. In fact, both circles are almost undistinguishable since E_0 and C are practically coincident $(C'E_0 \doteq 2$ cm$)$.

Due to the action of external forces, namely the attraction of sun and moon causing precession and nutation, H is different from C' and I is different from E_0. The angulum momentum axis H describes a circle (actually, a more complicated but known closed curve) around C', and I does the same around E_0. The period of the "free" motion of C' and E_0 is the Euler period of about 300 days, whereas the period of the "forced" motions of H around C' and of I around E_0 is about one day (since sun and moon remain almost in the same place when the earth performs one rotation); in the latter case, we speak of diurnal motions.

Some remarks about magnitudes: The radius of the "free" polar motion, $FC' \doteq FE_0$, is on the order of 6m , whereas the radius of the "forced" motion is smaller by one order of magnitude, namely $C'H \doteq E_0I \doteq 60$ cm. The points H and I are very close together, namely $HI \doteq 1.5$ cm; the same holds for $C'E_0 \doteq 2$ cm as we have already remarked.

Figure 7.1 shows that all points on the line FE_0 have no diurnal polar motion (i.e., forced body-referred motion due to external forces and having a nearly diurnal period). Of all points of this line, only the point C' has the property that its motion in space is computable independently of Chandlerian polar motion (i.e., free polar motion in the absence of external forces). It may be shown that this is equivalent to the fact

that C' has no diurnal nutation in space.

Regarding terminology it should be noted that body-referred periodic motions are called "polar motion" and space-referred periodic motion are called "nutation".

Elastic earth. In the more realistic case of an elastic earth, Fig. 7.1 must be modified as follows (Fig. 7.2). The angular momentum pole H and the instantaneous rotation pole I again describe circles (actually, more complicated closed curves) around C' and E_0, respectively; this is a forced motion (due to external forces); the radii of these circles are again around 1/2 m.   Due to the free Chandlerian motion, the points C' and E_0 again describe a radius on

Figure 7.2:   Body-referred motions for an elastic earth.

the order of 6m ; now, however, the center O of these circles no longer coincides with the axis of figure.

In fact, the non-coincidence of axis of figure in the Chandlerian motion causes an elastic deformation of the body, which causes the axis of figure (axis of maximum inertia) to deviate from the center O; it has the position S in Fig. 7.2 ($OS \doteq 2m$). The figure axis would be in S if there were only a free motion; due to the forced motion (attraction of sun and moon which also causes precession and nutation), the axis of figure F describes a circle (again, actually a more complicated closed curve) around S; the radius of this circle is very large, around 60 m. (It is easy to remember: $E_0 I \doteq 60$ cm,[1] $OC' \doteq OE_0 \doteq 6$ m, $SF = 60$ m !) Thus, S represents the mean position of F because of forced motion, and O is the mean position of S because of free motion.

It can be shown (Leick, 1978, p. 40) that F, H, and I lie on a common straight line.

Thus, the attraction of sun and moon cause the figure axis F to deviate by around 60 m from a mean position, whereas the same forces displace H and I only by 60 cm. This renders the axis of figure completely unsuited to serve as an axis of reference.

The periods of forced motion (of F, H, and I) are about 1 day, the period of free motion of S, C' and E_0 around O is the Chandler period of about 430 days.

Again, the point C' is distinguished by the fact that it has no diurnal motion, neither with respect to an earth-fixed system nor with respect to inertial space. It is, therefore, well suited to play the role of celestial pole.

More realistic earth models consider a liquid core (Melchior, 1978, Chapter 6). In this case, the celestial pole refers to the solid shell (it may then be denoted by C").

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[1] For an elastic earth, $C'H \doteq 0.7 E_0 I \doteq 40$ cm.

The present definition of the celestial pole is independent of any earth model and may even be used for the real earth: <u>The celestial pole C corresponds to the axis which has neither periodic diurnal body-fixed nor space-fixed motions.</u> The body-fixed motions of C are called polar motion; the space-fixed motions of C are precession and nutation.

This definition is even more natural in view of the fact that most astronomical measurements refer to this pole (Leick, 1978; Leick and Mueller, 1979; Kinoshita et al., 1979).

New IAU nutational theory. At its recent General Assembly in Montreal, August 1979, the IAU has adopted a new theory of nutation. It is computed for the "Celestial Ephemeris Pole" which is the celestial pole C as defined above. The earth model underlying the new nutational coefficients is Molodensky's Model II which features a solid inner and liquid outer core (cf. Melchior, 1978, pp. 153-156). More details can be found in (Seidelmann et al., 1979). An indication of the accuracy of this new theory is provided by the fact that, for the most recent earth model by Wahr (1979), the nutations differ from Molodensky's model by up to 0.002" (about 6 cm); cf. (Wahr and Smith, 1979).

8. Tidal Effects

The principal time variations which can be modeled in the sense of Section 2, are the solid earth tides.

The tidal potential, caused by the attraction of sun or moon, is (Melchior, 1978, p. 10)

$$U \;=\; G\mu \; \frac{r^2}{d^3} \; P_2(\cos z), \qquad\qquad (8\text{-}1)$$

where G is the gravitational constant, μ denotes the mass of the disturbing body (sun or moon), r is the radius vector of the point P at which U is considered, d is the distance between the geocenter and the disturbing body, P_2 is the Legendre's polynomial of second degree, z is the geocentric zenith distance of the disturbing body expressed by

$$\cos z = \cos \theta \sin \delta + \sin \theta \cos \delta \cos (h_G - \lambda) \quad . \tag{8-2}$$

Here θ and λ are polar distance and longitude of P, where

$$\theta = 90^\circ - \varphi \quad , \tag{8-3}$$

φ being the geographical latitude; δ is the declination of the disturbing body, and h_G is its hour angle reckoned from the zero meridian.

The addition theorem for Legendre polynomials (Heiskanen and Moritz, 1967, p. 33) gives

$$
\begin{aligned}
P_2 (\cos z) = \; & P_2 (\cos \overline{\delta}) \; P_2 (\cos \theta) + \\
& + \; 1/3 [R_{21}(\overline{\delta}, h_G) R_{21}(\theta, \lambda) + S_{21}(\overline{\delta}, h_G) S_{21} (\theta, \lambda)] \\
& + \; 1/12 \, [R_{22}(\overline{\delta}, h_G) R_{22}(\theta, \lambda) + S_{22} (\overline{\delta}, h_G) S_{22} (\theta, \lambda)],
\end{aligned}
\tag{8-4}
$$

where

$$
\begin{aligned}
R_{nm}(\theta, \lambda) &= P_{nm}(\cos \theta) \cos m \lambda, \\
S_{nm}(\theta, \lambda) &= P_{nm}(\cos \theta) \sin m \lambda,
\end{aligned}
\tag{8-5}
$$

P_{nm} denoting the Legendre function of degree n and order m. The $R_{nm}(\delta, h_G)$ and $S_{nm}(\overline{\delta}, h_G)$ are defined in the same way, with (θ, λ) replaced by $(\overline{\delta}, h_G)$; here

$$\overline{\delta} = 90^\circ - \delta$$

denotes the complement of declination (the polar distance) of the disturbing body.

Now (8-4) is substituted into (8-1) and $1/d^3$ and $R_{nm}(\delta, h_G)$ and $S_{nm}(\overline{\delta}, h_G)$ are represented as functions of time t using the theory of the motion of sun and moon. The result has the form

$$
\begin{aligned}
U = \; & a_1 R_{20}(\theta, \lambda) + a_2 R_{21}(\theta, \lambda) + a_3 S_{21}(\theta, \lambda) + \\
& + a_4 R_{22}(\theta, \lambda) + a_5 S_{22}(\theta, \lambda),
\end{aligned}
\tag{8-6}
$$

where the coefficients

$$a_i = a_i(t) \tag{8-7}$$

are functions of time, which may be represented as series of trigono-metric functions:

$$a_i(t) = a_{i0} + \sum_{j=1}^{\infty} a_{ij} \cos \omega_j t + \sum_{j=1}^{\infty} b_{ij} \sin \omega_j t . \tag{8-8}$$

Thus the tidal potential at the earth's surface, regarded as a sphere (r = const.), is given by 8-6) together with (8-8). It has the form of a linear combination of spherical harmonics of the second degree[1] whose coefficients are quasi-periodic functions of time; cf. (Melchior, 1978, Chapter 1).

Elastic deformation. In the case of a purely elastic earth, a point P on its surface undergoes a quasi-periodic displacement (on the order of ± 0.5 m) expressed by the displacement vector $\underline{u}$ whose components in polar coordinates $(r, \theta, \lambda,)$ are given by

$$u_r = \frac{h}{g} U ,$$

$$u_\theta = \frac{l}{g} \frac{\partial U}{\partial \theta} , \tag{8-9}$$

$$u_\lambda = \frac{l}{g} \frac{\partial U}{\cos \varphi \, \partial \lambda}$$

Here g denotes a mean value of gravity (g = 980 gal), and h and l are constants called Love numbers.

The deformation of the earth causes its gravitational potential to change by $\delta V = kU$, where k is a third Love number. In space, this

---

[1] The second degree is dominant in the tidal potential; higher-degree harmonics can be treated in the same way.

induced potential   is a harmonic function:

$$\delta V = k(\frac{R}{r})^3 [a_1(t)R_{20}(\theta,\lambda) +$$

$$+ a_2(t)R_{21}(\theta,\lambda) + a_3(t) S_{21}(\theta,\lambda) +$$

$$+ a_4(t)R_{22}(\theta,\lambda) + a_5(t) S_{22}(\theta,\lambda)].$$

(8-10)

Conventional average values are

$$h = 0.60, \quad k = 0.30, \quad \ell = 0.08 .$$

(8-11)

Variation of Love numbers with frequency.   The earth's liquid core produces resonance effects which render the Love numbers dependent on the frequency ω_i (Melchior, 1978, Chapter 6); we have now

$$h_j = h(\omega_j), \quad k_j = k(\omega_j), \quad \ell_j = \ell(\omega_j) .$$

(8-12)

Thus the expression (8-10) for the potential disturbance δV must be modified:

$$\delta V = (\frac{R}{r})^3 \sum_{i=1}^{5} [k_0 a_{i0} + \sum_{j=1}^{\infty} k_j(a_{ij}\cos \omega_j + b_{ij} \sin \omega_j t)] \cdot Y_i(\theta,\lambda),$$

(8-13)

where $Y_1(\theta,\lambda) = R_{20}(\theta,\lambda)$, $Y_2(\theta,\lambda) = R_{21}(\theta,\lambda)$, etc.   The displacement components (8-9) will be described by analogous expressions, with h_j and ℓ_j instead of k_j.

The constant k_0 in (8-12) deserves some discussion.   It corresponds to a permanent deformation independent of time.   In the case of a purely elastic earth, all k_j including k_0 are equal to the same constant $k \doteq 0.30$; see (8-11).   If the earth were fluid, then

$$k_0 = k_f \doteq 0.96,$$

(8-14)

which is called secular Love number   (Munk and MacDonald, 1960, pp. 25-26).   There is a marked contrast between the elastic Love

number 0.30 and the secular Love number 0.96. It is sometimes argued that the secular Love number (8-14) or some intermediate value between k and k_f should be used as k_0 in (8-13). For relevant discussions see (Munk and MacDonald, 1960, p. 27) and (Groten, 1970).

The modern earth models (Molodensky; Wahr, 1979) all give k_0 close to 0.30. Therefore, and for other reasons, the author thinks that the secular Love number (8-14) should not be used in tidal computations.

Computation of the tidal correction. If the displacement components (8-9) have been computed, then the corresponding tidal variations in the rectangular coordinates xyz are obtained by a rotation:

$$
\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\lambda & -\sin\lambda & \cos\varphi\cos\lambda \\ \sin\varphi\sin\lambda & \cos\lambda & \cos\varphi\sin\lambda \\ -\cos\varphi & 0 & \sin\varphi \end{bmatrix} \begin{bmatrix} u_\theta \\ u_\lambda \\ u_r \end{bmatrix}
$$

$$(8-15)$$

The corrected values of $\underline{x} = [x, y, z]$ are then obtained by

$$\underline{x} = \underline{x}_{measured} - \delta\underline{x} . \qquad (8-16)$$

The values $\underline{x}$ so obtained are theoretically free from the effect of tides; they should be constant in time, provided the model underlying the tidal corrections is adequate.

Why should the model not be adequate? The direct tidal potential (8-1) or (8-6) is independent of the internal structure of the earth and can, therefore, be calculated with a high degree of precision. The tidal response, that is, the deformation vector $\underline{u}$ and the induced potential δV depend, however, essentially on the earth's internal structure and physical properties. As we have seen, the Love numbers h, k, ℓ would be constant and independent of frequency if the earth were purely elastic; the liquid core makes the Love numbers dependent on frequency. They

can be calculated on the basis of some assumed earth model; an empirical determination, which would be independent of such earth model assumptions, is not at present feasible with the accuracy necessary to predict tidal effects on the centimeter level.

A further complication is caused by the fact that there are local disturbances due to interaction of solid tides (considered here) and ocean tides, the so-called ocean loading effects)(Melchior, 1978, Chapter 11), and other local perturbations (ibid., Chapter 12).

Even if these effects are taken into account computationally as well as possible, it does not seem possible to compute, on the basis of available tidal models, the tidal displacement of the position vector $\underline{x}$ to the desired accuracy of 1-2 cm. Errors may well reach magnitudes ten times as high.

A solution would be to monitor tidal effects on x, y, z at certain stations, obtaining observed values of δx, δy, δz and to compute corrections

$$\underline{v} = \delta \underline{x}_{observed} - \delta \underline{x}_{computed} ,$$

which can then be interpolated at other stations (E. Groten, personal communication).

One might also consider tidal models in which h and k depends, not only on frequency, but also on position (θ, λ) on the earth's surface as suggested by Kaula and (Groten, 1979, Vol. II).

In principle, h could also be obtained locally by a combination of gravimetric observations, which give $1 + h - 3k/2$, and horizontal pendulums, which give $1 + k-h$; the Love number ℓ can be found by extensometer measurements. If the values of h and ℓ so obtained were sufficiently accurate to be used for modelling the displacement vector $\underline{u}$, this procedure might be simpler than the monitoring of x, y, z themselves.

The Honkasalo term. The constant term a_{i0} in (8-8) is
independent of time and causes a "permanent deformation". Actually,
only the coefficient a_{10} associated with the zonal harmonic $R_{20}(\theta,\lambda) =$
$P_2(\cos\theta)$ is different from zero, increasing, so to speak, the flattening
of the earth. It has been suggested by Honkasalo (1964) to correct
only for the time-dependent part of the tidal effects, leaving the
permanent deformation.

This innocently looking procedure has, however, far reaching
consequences as pointed out by Heikkinen (1979). In fact, it means
that, e.g., in the case of the moon, the lunar attraction is not
completely removed but a residual effect is left which can be physically
interpreted by uniformly distributing the moon's mass along the lunar
orbit.[1] This means, however, that sun and moon are not completely
removed computationally, so that the external gravitational potential
is no longer a harmonic function in outer space as presupposed in the
usual methods of physical geodesy (the level ellipsoid, Stokes' formula,
Molodensky's theory, least-squares collocation, etc.).

For this reason it is much simpler to remove the total tidal
effect, including the permanent deformation, which corresponds to the
application of formulas such as (8-1) and (8-6) together with (8-8). If
the permanent deformation is not removed, matters become so compli-
cated that confusions are almost inevitable.

At any rate, an official decision of the IAG on this point is highly
desirable and is expected at the IUGG General Assembly in Canberra,
December 1979.

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[1] More precisely, the moon's orbit must be projected onto the equa-
torial plane and replaced by an average circle, on which the moon's
mass is then distributed.

9. Terrestrial Reference Systems

As we have already remarked at the end of Section 2, the problem of introducing an appropriate, rigorously defined terrestrial coordinate system is complicated by the fact that there is no such system in which all points of the earth's surface would be at rest. Points are continuously moving because of tidal effects, plate motion, local tectonic disturbances, etc. All that can be hoped for is that a coordinate system can be defined at which the stations are at rest in some average way.

This problem can certainly be solved, but there are many possible solutions. At present, there is considerable controversy in this respect, as the proceedings of a recent meeting on this subject show (Kolaczek and Weiffenbach, 1975).

It is, however, generally agreed that even a future, more precisely defined, reference system should be close to the presently used system. This means that the origin should coincide with the geocenter (the earth's center of mass), or correspond to some average position of the geocenter; the z-axis should be directed along some average position of the rotation axis, and the zero meridian should be close to the mean Greenwich meridian.

The origin. The geocenter is uniquely defined physically as the center of mass of the earth. It is indirectly accessible to observation using one or the other of two physical phenomena:

1) The geocenter is at a focus of the orbital ellipse of a satellite (perturbations of satellite orbits do not essentially change the picture).

2) If the geocenter is made to be the origin of a spherical-harmonic expansion of the earth's gravitational potential, then all first-degree terms of this expansion vanish (Heiskanen and Moritz, 1967, p. 62).

For a precise <u>absolute</u> practical determination of the geocenter, the first phenomenon is appropriate. The accuracy depends on the precision with which the orbit is practically defined.

The second phenomenon provides an excellent means for monitoring changes in the geocenter, as has been pointed out by Mather (1973) and (Mather et al., 1977). At any point (θ, λ) the earth's surface, gravity changes because of a shift $\delta\underline{x}$ of the origin by

$$\delta g = c(\delta x \cos\varphi \cos\lambda + \delta y \cos\varphi \sin\lambda + \delta z \sin\varphi) \qquad (9-1)$$

where

$$c = -3.08 \, \mu \, \text{gal cm}^{-1}. \qquad (9-2)$$

Although three stations would, in principle, sufficient to determine δx, δy, δz from observed changes δg, a larger number of well-distributed observatories, continuously observing absolute g, is necessary to separate geocenter motion from other effects. Mather advocates a minimum of 25 globally distributed absolute gravity stations.

<u>Various definition of coordinate axes.</u>   Various choices of axes have distinguished physical properties, as pointed out in (Munk and MacDonald, 1960, pp. 10-12).

<u>Tisserand axes.</u> For a rigid body rotating with the angular velocity vector $\underline{\omega}$, the velocity of any particle is

$$\underline{v} = \underline{\omega} \times \underline{x} \qquad (9-3)$$

which is the vector product of $\underline{\omega}$ with the position vector $\underline{x}$. For a deformable body this relation cannot be satisfied in general; there will be a difference

$$\underline{\epsilon} = \underline{v} - \underline{\omega} \times \underline{x} \; . \qquad (9-4)$$

If $\underline{\omega}$ is defined in such a way that

$$\iiint_{\text{earth}} \underline{\epsilon}^2 \, dM = \text{minimum}, \qquad (9-5)$$

dM being the mass element, then any system xyz rotating with this angular velocity $\underline{\omega}$ is a system of Tisserand axes.

The minimum property (9-5) is, of course, some kind of least-squares condition and should, therefore, have a certain appeal for geodesists.

Such a system also has the property that the total angular momentum due to motion relative to the system is zero.

It should be noted that in the case of Tisserand axis, only the rotation of the frame, that is, its motion, is specified. The direction of the axes is arbitrary in the sense that a system whose axes are oriented in a fixed way relative to a Tisserand system (differ from it by a constant rotation) is also a Tisserand system.

A Tisserand frame is very suitable for formulating the equations of motion for the earth, the Liouville equations, because they assume a particularly simple form in such a system: they are then, for a deformable body, formally the same equations as for a rigid body. Disadvantages are that wind and other relative motion may slightly rotate the Tisserand axes relative to observatories, and it is difficult to formulate mathematically relative motion in such a frame.

The exact practical realization of a Tisserand frame for the real earth seems hardly feasible. It should, however, be pointed out that, for the elastic model discussed in Section 7, the point O in Fig. 7.2 (as well as any other point rigidly connected to it) corresponds to the z-axis of a Tisserand frame. Since this point is not subject to polar motion, it is a natural choice for this axis. For the real earth, the z-axis should, therefore, be selected close to such a point, and the xyz system should be close to a Tisserand frame.

Principal axes of inertia. These axes defined in such a way that the inertia tensor is diagonal in this system; the products of inertia are then zero (cf. Heiskanen and Moritz, 1967, p. 62). They are

natural generalizations of axes of symmetry (e.g. for the triaxial ellipsoid) to an arbitrary body and are, therefore, also called figure axes.

In the case of the earth, the equatorial principal axes (x and y) are ill defined since the earth is very close to an ellipsoid of revolution.

Even the polar axis of inertia is unsuited for a precise definition of the z-axis because of its instability: the point F, to which it corresponds, oscillates with respect to the earth by as much as 60 m (Section 7).

However, as Fig. 7.2 shows, the point O corresponds to a mean position of the figure axis; it is also a Tisserand axis. An axis close to this point will thus also be close to a mean figure axis.

If the z-axis coincides with the (mean) figure axis, then the spherical-harmonic coefficients of degree 2 and order 1 must vanish (on the average). Since actual satellite and gravity observations will correspond to such an average, it will be reasonable to enforce the condition that the two coefficients of degree 2 and order 1 vanish. In this way it will be ensured that the z-axis coincides approximately with the mean polar figure axis. It is clear that the accuracy of such a procedure is low (around 10^{-3}) so that a definition precise to 10^{-8} cannot be achieved in this way.

Mather axes. In his thorough discussion of precise terrestrial reference systems, Mather (1973) proposed the following definition. The origin is at the (instantaneous) geocenter; the z-axis coincides with the instantaneous axis of rotation; and one fixed station P on the earth's surface determines the xz-plane (this plane either passes through P ro P has an assigned fixed longitude).

This is perhaps the conceptually clearest and most natural definition of a geodetic reference system. Everything -- the origin the z-axis, and the xz-plane -- is unambiguously defined physically. Its main merit lies in presenting a clear theoretical model. For practical

purposes it appears less suited since the instantaneous rotation axis I is not fixed within the earth's body (Section 7). Thus, even if the earth were rigid, the coordinates xyz of any point of the earth's surface would undergo periodic changes which would not correspond to any movement of the point but merely reflect the variation of the coordinate axes.

The identification of the z-axis with the celestial pole C, rather than with the instantaneous pole I, would remove the diurnal variations in the coordinate system but still leave its variation due to polar motion. Also, C is observable rather than I (Section 7).

Therefore, the z-axis should rather be directed along a mean position of C, which would be the point denoted by O in Fig. 7.2. This point denotes at the same time a mean axis of figure and a mean Tisserand axis as we have seen.

As for the point P held fixed, it is clear that tidal motions have to be removed. A difficulty remains: residual unknown motions (plate motions, local tectonic displacements) are reflected as spurious changes in the coordinates of all terrestrial points. It appears, therefore, desirable to define the coordinate system with respect, not to one point, but to several reference points, hoping that irregular displacements of individual points somehow average out. This leads us to:

Geographical axes. According to (Munk and MacDonald, 1960, p. 11), geographical axes are attached "in a prescribed way" to certain observatories. A rigorous definition in this sense would be the following.

Assume N stations (observatories) on the earth's surface. The coordinates $\underline{x}_i = (x_i, y_i, z_i)$, $i = 1, 2, \ldots, N$, of these stations, referred to a certain epoch t_0, are given; the coordinate system S_0 is, in principle, arbitrary.

At a subsequent epoch, say t_1, the rectangular coordinates of the same N stations are again determined by observation; this gives the values $\underline{x}_i' = (x_i', y_i', z_i')$. They are referred to a coordinate system

S' which is not the same as the former system S_0. The system S' is assumed to be arbitrary and unrelated to S_0.

If the configuration of the N stations did not change with time, then there would be a certain rotation matrix $\underline{R}$ such that

$$\underline{R}\underline{x}' = \underline{x} . \tag{9-6}$$

for all N stations.

In view of relative motion of the stations, however, such an equation will not be exactly satisfied; there will be deviations

$$\underline{\epsilon} = \underline{R}\underline{x}' - \underline{x} . \tag{9-7}$$

Now the three parameters defining the rotation matrix $\underline{R}$ (for instance, three Eulerian angles) can be determined by means of the condition

$$\underline{\epsilon}^T\underline{P}\underline{\epsilon} = \text{minimum} \tag{9-8}$$

with a given positive definite weight matrix $\underline{P}$. [1]

This determines, $\underline{R}$, and now the coordinates $\underline{x}$ of any point in the new system S' can be transformed to the original system S_0 by (9-6). For the given stations, the comparison of the original with the transformed new coordinates will indicate the amount by which the stations have moved (assuming there are no observational errors); and the coordinates of other points are in this way transformed into the original system S_0.

Thus, the coordinates at any instant t can be unambiguously related to the original system S_0. The coordinate system is related, not to any physically defined axes, but "in a prescribed way" to the N given observatories.

In the case of errorless observations, the formal "least-squares adjustment" by means of (9-8) does nothing else but ensure that the

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[1] The minimum condition (9-8) is, in a way, a discrete analogue to the Tisserand condition (9-5).

configuration of the N stations at epoch t_1 is fitted as closely as possible to the original configuration of these stations. The "prescribed way" implies the use of the same N stations and of the same matrix $\underline{P}$ at all epochs under consideration.

It is clear that tidal effects and other systematic phenomena which can be modeled have to be removed; the least-squares fit is made for the residuals $\delta x(t)$ in the notation of eq. (2-6).

If there are observational errors, then, of course, the procedure averages their effect as well as that of actual displacement. The weight matrix P may then be chosen such as to take into account statistical characteristics of the measuring errors.

Each of these four definitions -- Tisserand axes, figure axes, Mather axes, and geographical axes -- contain important aspects which must be incorporated in an optimal definition of a terrestrial reference system.

Geographical axes seem best to correspond to the practical requirement that the adopted station coordinates do not change with time. _They can also be realized observationally in a theoretically rigorous way.

The arbitrariness of the initial coordinate system S_0 can be used to relate the z-axis to the earth's rotation axis. For the elastic earth, the point O in Fig. 7.2 represents the long-term average of the rotation axis, of the celestial pole, and of the figure axis; it is, furthermore, a Tisserand axis as we have remarked above.

For the real earth, there is no longer a unique point O which has all these physical properties. Therefore, the z-axis will be defined conventionally in such a way as to be close to a mean rotation axis and a mean figure axis. If it turns out that the geocenter shifts significantly with time, then the origin will also have a conventional position (defined by the N reference stations) close to the geocenter.

Thus, the system will be a <u>conventional</u> system close to a <u>physically</u> defined one; it is also an <u>average</u> system in view of condition (9-8). This relates the present concept of a terrestrial reference system to the ideas presented in Section 2.

<u>Present system.</u> The present internationally adopted system is the BIH system described by Guinot (1978). The Conventional International Origin (CIO) for polar motion, adopted in 1967, which defines the z-axis of the terrestrial coordinate system, is based on the continuous (since 1899) observations of the 5 stations of the ILS. The CIO approximately corresponds to the mean pole during the period 1900-1905. The ILS observations, through less accurate (around 1 m) than modern methods for polar motion determination, provide long-term stability.

The basic equations for polar motion are

$$\Phi = \Phi_{obs} - x \cos \Lambda + y \sin \Lambda, \qquad (9 - 9)$$

$$\Lambda = \Lambda_{obs} - (x \sin \Lambda + y \cos \Lambda) \tan \Phi, \qquad (9 - 10)$$

where Φ and Λ are astronomical latitude and longitude referred to the basic terrestrial system xyz, Φ_{obs} and Λ_{obs} are the observed values referred to the instantaneous celestial pole C, and x and y are the coordinates of the pole referred to CIO, not to be confused with rectangular coordinates xyz (Section 6).

The method of determining polar motion used by ILS consists in keeping Φ for the five latitude stations fixed, assigning to them conventional values. Then each of these stations gives an observation equation of form (9-9). A least-squares adjustment of these 5 equations for x and y then determines these two polar coordinates; cf. (Fedorov, 1979, p. 96).

If is evident that this procedure is rather analogous to that represented by (9-7) and (9-8), with the geographical latitude Φ instead of cartesian coordinates xyz.

Similarly, the BIH zero meridian represents an average over about 50 time service stations (Mueller, 1969, p. 343) and thus provides an analogue to the minimum principle (9-8) but applied to observed longitude Λ (or, equivalently, Universal Time).

Since 1962, when the ILS was reorganized into the IPMS, many more latitude stations (around 50) contribute to defining the pole than before. The accuracy is now on the order of 0.2 m, about 3 times better than for the ILS (Guinot, 1978, p. 14).

We may thus say that, from a conceptual point of view, the present BIH system represents geographical axes as defined above, but with the least-squares fitting applied to Φ, Λ rather than to x, y, z.

Future requirements. For a more precise definition, at the centimeter level, satellite laser and interferometric methods determining cartesian coordinates x, y, z seem to be better suited than astronomical coordinates Φ, Λ whose accuracy can hardly be essentially improved beyond the present level.

The best approach seems to be that systematic effects (tides, plate motion, etc.) should be modeled as far as possible, and that residual, more or less random, motions and similar effects should be taken into account by averaging over a certain number of given stations. This corresponds to the procedure described above, eqs. (9-7) and (9-8).

For a consistent definition it would be desirable that one uses always the same N stations, the same observation techniques, and the same weight matrix $\underline{P}$.

The system thus defined is conventional in the sense that the underlying coordinate system S_0 as defined above is, in principle, arbitrary.

It should, however, be related to the present system in such a way as to preserve continuity. A link between a cartesian system S_0 and the presently used Φ, Λ- system is provided by the determination of polar motion by modern methods such as doppler, laser, and VLBI which are related to such a cartesian system.

The system S_0 should be made geocentric by incorporating dynamic satellite observations. To be precise, it should be geocentric at the initial epochs t_0. The possibility cannot be excluded that, at later epochs, the geocenter has slightly shifted with respect to the frame S_0 which continues to be defined by the N basic stations. Such a shift can be monitored by absolute gravity measurements as mentioned above.

The basic terrestrial cartesian frame also provides the orientation of the reference ellipsoid (IAG, 1970; Moritz, 1975, 1979).

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